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DEVELOPMENT OF A METHOD FOR COMPUTER SIMULATION OF A SWINGING SPRING LOAD MOVEMENT PATH

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Продовжено дослідження геометричного моделювання нехаотичних періодичних траєкторій руху вантажів різновидів математичних маятників. Розглядаються маятникові коливання у вертикальній площині підвішеної невагомої пружини, зберігаючої при цьому прямолінійність своєї осі. В літературі такий вид маятника називають хитною пружиною (swinging spring). Шукана траєкторія вантажу хитної пружини за допомогою комп'ютера моделюється з використанням значень маси вантажу, жорсткості пружини та її довжини в ненавантаженому стані. Крім того, використовуються такі початкові величини параметрів ініціювання коливань хитної пружини: кут відхилення осі пружини від вертикалі, швидкість зміни величини цього кута, а також параметр подовження пружини та швидкість зміни подовження. Розрахунки виконано за допомогою рівняння Лагранжа другого роду. Також розглянуто варіанти знаходження періодичних траєкторій точкового вантажу хитної пружини з рухомою (вздовж координатних осей) точкою кріплення.

Актуальність теми визначається необхідністю дослідження та удосконалення нових технологічних схем механічних пристроїв, до складу яких входять пружини. Зокрема, дослідження умов відмежування від хаотичних коливань елементів механічних конструкцій та визначення раціональних значень параметрів для забезпечення періодичних траєкторій їх коливань.

Наведено спосіб знаходження значень набору параметрів для забезпечення нехаотичної періодичної траєкторії руху точкового вантажу хитної пружини. Ідею способу пояснено на прикладі знаходження періодичної траєкторії руху другого вантажу подвійного маятника.

Наведено варіанти розрахунків для одержання періодичних траєкторії руху вантажу, коли задані параметри:

– жорсткість пружини та її довжина без навантаження, але невідома величина маси вантажу;

– величина маси вантажу та довжина пружини без навантаження, але невідома жорсткість пружини;

– величина маси вантажу та жорсткість пружини, але невідома довжина пружини без навантаження.

Також розглянуто знаходження значень набору параметрів для забезпечення умовно періодичної траєкторії руху точкового вантажу хитної пружини з рухомою точкою кріплення.

Побудовано фазові траєкторії функцій узагальнених координат (значень кутів відхилення осі пружини від вертикалі та подовження хитної пружини) за допомогою яких можна оцінити діапазони зазначених величин та швидкостей їх зміни.

Результати можна використати як парадигму для вивчення нелінійних зв'язаних систем, а також при розрахунках варіантів механічних пристроїв, де пружини впливають на коливання їх елементів. Коли в технологіях використання механічних пристроїв необхідно відмежуватися від хаотичних переміщень вантажів, а забезпечити періодичні траєкторії їх руху

Ключові слова: маятникові коливання, періодичної траєкторії руху, хитна пружина, рівняння Лагранжа другого роду

1. Introduction

In order to explain complex processes occurring in nature, pictorial mechanical interpretations are often used. Specifically, pendulum analogs are used as models of oscillatory processes [1].

A model of inverse pendulum with oscillating fixing point is a classic example. Physical model of this pendulum is the basis of theory of dynamic stabilization. The key idea of the theory consists in the need of dividing movement into “fast” and “slow” components, which was reflected in the concept of effective potential. With the help of the method of effective potential, the principle of stability of high-frequency generator, “nigotroneum”, was explained [2]. Apropos, in order to have no confidentiality problems when publishing the method, a physical model of a pendulum with oscillating suspension was taken, which would illustrate the principle of generator stability. This has served as a starting point for mathematical study of the pendulum with an oscillating suspension.

No less impressive mechanical interpretations are associated with a pendulum of another kind. In an idealized form, pendulum is a vertically suspended weightless spring with a point load attached at its end. In addition to longitudinal oscillations, spring oscillates like pendulum in a vertical plane while maintaining straightness of its axis. It was noted that if the load simultaneously performs oscillations along the spring axis and pendulum oscillations, then this action demonstrates the phenomenon of spring oscillations from a completely unexpected side. Behavior of such vibratory system has revealed interesting and deep physical laws [3].

The model of a spring oscillating like a pendulum (it is called a swinging spring in literature) is widely used as a mechanical model of more complex processes in nature and technology. These are processes with internal, nonlinearly coupled systems providing various oscillating components. What is essential in this process is the fact that the system components perform energy exchange with each other. Analysis of such energy exchange processes is presented in [1] in order to find out how all this depends on the system control parameters. To illustrate this, authors use a swinging spring as a paradigm for studying nonlinear coupled systems. Three energy components are identified for a swinging spring. They are similar to the movements of a spring, a pendulum and a link between them. The presented procedure can, in principle, be applied to arbitrary nonlinear coupled systems to show how the link mediates internal energy exchange processes and how energy distribution varies according to the system parameters.

The feature of the swinging spring phenomenon can be illustrated graphically. To this end, compare movement paths of the point load for two cases: a swinging spring (Fig. 1, *a*) and a parametric pendulum (Fig. 1, *b*).

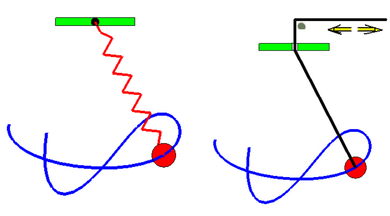


Fig. 1. Analogy between angular oscillations: swinging spring (*a*); mathematical parametric pendulum (*b*)

For a parametric pendulum, parameter effect is manifested by the change of the pendulum length due to an external energy source. There is an interesting case when length is to be slightly increased in a low position and slightly reduced in extreme positions. Then maximum swaying will be achieved when frequency of the system parameter change (suspension length) is twice that of the system’s own oscillation frequency, for example, swinging of a children’s swing. To maintain its swinging for a long time, it is necessary to squat quickly at the moment of the greatest deviation of the swing from the position of equilibrium and stand up quickly when passing the lower position.

However, there is a fundamental difference between the “swinging spring” pendulum and the “swing” pendulum. There is no external energy source in the swinging spring and pendulums of this kind must themselves “provide” existence of such oscillations. It follows from experiments that growth of angular oscillations of the swinging spring is accompanied by attenuation of longitudinal oscillations. Then a reverse phenomenon takes place: swinging of longitudinal oscillations by means of reducing energy of angular oscillations. Further, the whole process is constantly repeating. Repetitive sequential energy pumping from one oscillation to another occurs until all oscillations extinguish because of friction.

Nonlinear coupled systems with interacting subsystems are present in many fields: from physics and engineering to biology and social sciences. Examples of coupled systems include wave unification in plasma physics, laser pumping, biological oscillatory nets, neural nets, and genetic chains (corresponding references are given in [1]).

Study of features of the swinging spring oscillations is of interest for practical applications. For example, study of atmospheric balance of the planet is carried out with the help of a swinging spring model in [4], carbon dioxide molecule oscillation is studied in [5], oscillation of high-voltage wires is considered in [6] and helicopter vibration is modeled in [7]. Description of spring oscillation is similar to equations from “predator-prey” problems [8]. The list can be continued. At the same time, seaming disparate at first glance implementations have a common feature: possibility of their study on the basis of the swinging spring model. At the same time, the key point is determination of conditions for ensuring non-chaotic periodic paths of the swinging spring load. Such studies make it possible to isolate from chaotic movements of elements of mechanical devices which include spring elements. Periodic path of movement of a swinging spring load illustrates solution of corresponding differential equations describing its oscillations. After all, these equations have a nature similar to that of differential equations of adjacent (by their subject matter) implementations. The resulting geometrical form of the periodic path of movement of a swinging spring load in the space of parameters of a specific task will help illustrate solutions to this problem. That is, consideration of the model of a swinging spring will allow one to analyze nature of solutions in adjacent by their subject matter tasks and identify optimal in a certain sense options among them, just like Lissage figures are used in mechanics for analysis of oscillatory processes in mechanisms.

Consequently, the relevance of the chosen topic indicates necessity of developing an engineering method for finding values of a set of parameters to provide non-chaotic periodic path of movement of a swinging spring point load.

2. Literature review and problem statement

History of the studies dedicated to oscillation of a swinging spring began with the quantum-mechanical explanation of the effect of line splitting in the spectrum of Raman scattering on C_2 molecule. At the same time, it was suggested that the effect is of not quantum but classical mechanical oscillation nature. Namely, the effect is due to internal peculiarities of the molecule oscillation where frequency of oscillations of one type is approximately twice frequency of oscillations of another type. Scientists have decided to test it on a swinging spring model. Calculated movement of such a system has shown that complete energy pumping from vertical oscillation to horizontal one and backward should periodically occur at a frequency ratio of 2:1.

Expediency of the swinging spring studies has arisen in a connection with the revealed possibility of their "non-standard" use both in theory and practice. However, most studies are focused on analytic approximations for weakly coupled systems and energy exchanges arising when subsystems resonate. Parametric mechanism is an effective mechanism of energy exchanges [9]. Specifically, a swinging spring with two degrees of freedom is an auto-parametric system being the basis for studying nonlinear coupled systems. In addition, swinging spring is important due to the possibility of qualitative presentation of many nonlinear coupled systems. Among these presentations, classical analog of vibrational modes of triatomic molecules can be mentioned. It realizes Fermi resonance in infrared and combinatorial spectra [1].

Oscillations of swinging springs are directly related to plane and ship dynamics. Effects of impairment of stability and controllability of high-speed ships and supersonic aircraft were revealed. It turned out that the most intense swinging of lateral oscillation occurs when incidence oscillation occurs at twice the frequency of lateral oscillation [10]. Flexible thread model as a modified swinging spring model plays an important role in building mechanics. Flexible thread is a kind of spring acting only in stretching. In a typical two-dimensional model, flexible thread can simultaneously perform transverse oscillations in its plane (analogous to angular oscillations of a swinging spring with a load attached) and pendulum oscillations that connect supports (analogous to vertical oscillations) [6, 11]. Loss of dynamic stability occurs at a ratio of frequencies of these oscillations 1:2 when transverse oscillation of the thread at amplitudes reaching rather large values arises. The possibility of occurrence of such phenomena must be taken into consideration in calculation of various structures (suspension bridges, cable-and-beam systems, cable-ways, power lines, various spaceship cable systems for holding objects, flexible hoses, various antennas, etc.) [3].

Theoretical study of small planar nonlinear oscillations of a swinging spring with a nonlinear dependence of its tension on lengthening is given in [12, 13]. The method of a Hamiltonian normal form was used. Solution of Hamiltonian equations of normal form has shown that periodic reorganization of oscillations between vertical and horizontal modes occurs only in the case of resonance ratios of 1:1 and 2:1. In all other cases, both in presence of resonance and in its absence, oscillations occur at two constant frequencies.

Changes in behavior of a swinging spring when one response under parameter checkout becomes unstable and is replaced by another are studied in [14]. Poincare sampling is used to reduce the problem of describing the limit-cycle stability to a simpler problem of determining stability of a fixed point by the Poincaré mapping. Connection of normal modes of a swinging pendulum oscillation is considered in [15]. Comments on experiments related to violation of normal modes are given. of a Swinging spring systems near resonance are investigated in [16] with the help of "slow fluctuation" approximation which consists in application of trigonometric polynomials and preservation of only a member with the slowest frequency. It was shown in [17] that integral approximation of a spatial swinging spring adjusted to a resonance of 1:1:2 has monochromium and a stepwise angle of precession of the plane of oscillation of the resonant spring pendulum is the number of revolution of integral approximation. Paper [18] is devoted to oscillation of a swinging pendulum with its suspension point moving along vertical line. Periodic solutions of the equation are obtained with the use of Hill's determinants. The developed computational procedure is used to determine combinations of those system parameters for which periodic solutions are possible. A spatial swinging spring with resonance of 2:1:1 approximately described by Lagrangian is studied in [19]. Hamiltonian abbreviations and sampling methods are used in descriptions. The resulting formula describes stepwise azimuthal angle precession. Energy flow between longitudinal and pendulum oscillations is considered in [20] as pulsation. Pulsation and stepped precession are characteristic features of the swinging spring dynamics. Hamiltonian reduction was used to find complete analytical solution. Dynamics of a spring pendulum is investigated in [21] with the use of asymptotic methods. Methods of the theory of nonlinear normal forms of oscillation have enabled study of pendulum dynamics both for significant and small oscillation amplitudes.

However, all of these studies are mostly theoretical. For engineering practice, methods are needed to construct real non-chaotic periodic paths of the swinging spring loads. Some of them are also described in [22] where examples of periodic paths are given as well as in [23] where conditions for constructing periodic paths are studied. A program written in Mathematics language by means of which periodic paths of a double pendulum can be constructed is presented in [24]. The study [25] is devoted to a connection of a possible spring load path with Lissage figures. A maple program for constructing spring load paths is presented in [26]. Another method of constructing spring load paths is proposed in [27]. Examples of periodic paths of swinging springs are given in [28]. Oscillation of a swinging spring with a moving suspension point is studied in [29]. However, there is no universal approach to construction of periodic paths of spring load in the known studies. Also, there is no oscillation analysis using phase paths of the functions included in description of generalized coordinates of a corresponding oscillatory system.

A method of projection focusing is presented in [30] for construction of periodic paths of loads of a variety of mathematical pendulums. Examples of implementation of this method are considered in [31].

As a result of the review of published sources [1–29], issues that have not yet been investigated by other authors were identified. They have enabled formulation of the following study problem: develop a method for finding values of a set of parameters that would provide a non-chaotic periodic path of a point load of a swinging spring, that is, a load attached to a vertically suspended spring performing pendulum oscillations.

3. The aim and objectives of the study

The study objective was to develop a method for computer modeling of a periodic path of movement of a point load attached to a swinging spring.

To achieve this objective, the following tasks had to be solved:

– explain the method idea on an example of a test task: construct periodic path of movement of the second load of a double pendulum;

– provide variants of calculation for obtaining periodic paths of movement of a swinging spring load when the following is given:

– length without load and stiffness at unknown load weight;

– spring length without load and the load weight at unknown spring stiffness;

– load weight and spring stiffness at unknown length of nonloaded spring;

– define a set of parameters for provision of a conditionally periodic path of movement of a point load attached to a swinging spring with a moving fixing point;

– construct phase paths of functions of generalized coordinates of the swinging spring (values of angles of the spring axis deviation from the vertical and elongation) in order to assess the range of variation of above quantities and the rate of this variation;

– illustrate the results obtained by computer animation of oscillation of corresponding swinging springs.

4. Developing a geometric model of periodic paths of a swinging spring load

4. 1. Constructing periodic paths of movement of a double pendulum load by the method of projection focusing

The method of "projection focusing" intended for construction of periodic paths of loads of a variety of pendulums is considered in [30]. Hereinafter, an explanation of the method is given by the example of determining non-chaotic path of movement of the second load attached to a double pendulum. This example is considered in most textbooks on theoretical mechanics. Namely double pendulum is so often used to illustrate chaotic oscillations. Therefore, solution to the problem of periodic paths of the second load on a double pendulum will be of particular interest.

Let us set conditions of idealization of oscillations of a double pendulum:

– both links are weightless and nondeforming;

– weight of the loads is concentrated in corresponding points at the ends of the links;

– there are no nodal supports and air resistance during oscillation;

– oscillation proceeds in a vertical plane enveloping the suspension point;

– the process of energy dissipation is slow in comparison with characteristic time scales (the oscillatory system is conservative);

– parameters and initial conditions are set in conventional numerical units.

The double pendulum diagram is shown in Fig. 2.

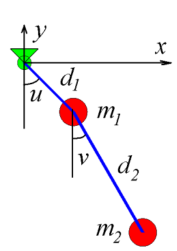


Fig. 2. The double pendulum diagram

Take the angles formed by the pendulum links with the vertical axis *Oy* as generalized coordinates $u(t)$ and $v(t)$. Then virtual coordinates of the nodal points can be calculated by formulas:

$$x_1 = d_1 \sin u; \quad y_1 = -d_1 \cos u; \tag{1}$$

$$x_2 = d_1 \sin u + d_2 \sin v; \quad y_2 = -d_1 \cos u - d_2 \cos v.$$

Set Lagrangian as a difference between kinetic and potential energies (g=9.81):

$$L = 0.5 m_1 \left(\frac{dx_1}{dt}\right)^2 + \left(\frac{dy_1}{dt}\right)^2 +$$

$$0.5 m_2 \left(\frac{dx_2}{dt}\right)^2 + \left(\frac{dy_2}{dt}\right)^2 - m_1 g y_1 - m_2 g y_2. \tag{2}$$

To compile a system of Lagrange differential equations of the second kind, use the following relations (the point here means time derivative):

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{u}}\right) - \frac{\partial L}{\partial u} = 0; \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{v}}\right) - \frac{\partial L}{\partial v} = 0. \tag{3}$$

As a result, the system of Lagrange equations of the second kind is obtained in the form:

$$(m_1 + m_2) d_1 \frac{d^2 u}{dt^2} + m_2 d_2 \frac{d^2 v}{dt^2} \cos(u - v) +$$

$$m_2 d_2 \left(\frac{dv}{dt}\right)^2 \sin(u - v) + (m_1 + m_2) g \sin u = 0; \tag{4}$$

$$d_2 \frac{d^2 v}{dt^2} + d_1 \frac{d^2 u}{dt^2} \cos(u - v) -$$

$$-d_1 \left(\frac{du}{dt}\right)^2 \sin(u - v) + g \sin v = 0.$$

To illustrate possibilities of the method of projection focusing, consider construction of a periodic path of the second load of a double pendulum in a form of the following problem.

Problem statement. Construct a periodic path of a double pendulum with link lengths d_1 and d_2 and weights m_1 and m_2 as loads at the ends of the links. In the initial position, all links are vertical, that is, $u(0)=0$ and $v(0)=0$. Oscillations are initiated using impulses applied to the pendulum loads in two mutually antithetic directions along the *Ox* axis (Fig. 3). That is, $du(0)=-F$ and $dv(0)=F$ where F is a quantity that can be characterized as the initial rate of change in time of the corresponding angle magnitude.

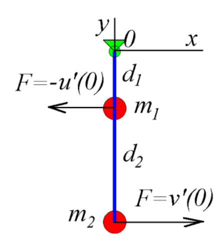


Fig. 3. Initiation of a double pendulum oscillations

Problem solution with simultaneous explanation of the method of projection focusing. For determination of the value of parameter F which would provide periodic movement path of the second load of the double pendulum, apply the method of projection focusing [30]. To do this, solve the system of equations (3) with chosen initial conditions $u(0)=0$; $du(0)=-F$; $v(0)=0$; $dv(0)=F$ and parameters using numerical Runge-Kutta method. "Assign" the variable F as a control parameter of oscillation of the double pendulum.

Next, construct image of the integral curve in the phase space $\{u, Du, t\}$ depending on the value of the control parameter F. At arbitrary values of F in the phase space, a "tangled" integral curve will more likely be formed (Fig. 4, *a*). In the algorithmic implementation, this will be a multi-link curved line which will connect N adjacent points with coordinates (u_i, Du_i, t_i) (where $i=1...N$). Points in the phase space are obtained as a result of numerical solution of the system of Lagrange equations of the second kind.

Project the resulting integral curve to the phase plane $\{u, Du\}$ where the phase path of the generalized coordinate function u(t) will be its projection (the same can be done for the coordinate function v(t)). When the control parameter F changes, character of the phase path also changes. At a certain critical value of $F=F_0$, character of the phase path will change at a qualitative level: it will turn into a "focused" curve. In the process of movement of the parameter F to the critical value F_0 in a mode of computer animation, one can observe an optical effect of "sharpening" of the "tangled" phase paths in the phase plane (Fig. 4, *b*).

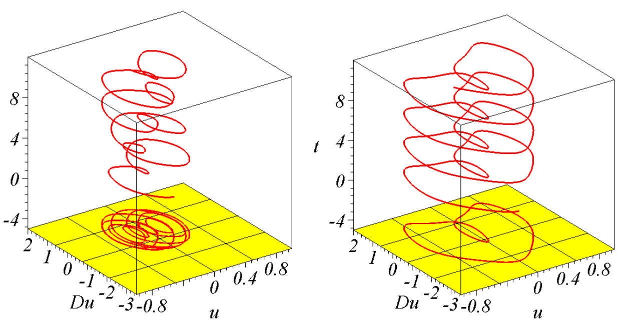


Fig. 4. Phase paths as projections of integral curves: for an arbitrary value of the control parameter F (*a*); for a critical value F_0 of the control parameter (*b*)

To correlate graphic properties of the phase curves with numerical ones, the concept of saturation (or density) of a line image was used. The property of saturation of the line image is characterized by the number of conditional plane points (pixels) built by means of computer graphics. The number of pixels to make line pictorial with an acceptable for practice error will be the degree of saturation. The value of the control parameter F is sought so that the image of the phase paths was of a minimum saturation which can be compared with the above-mentioned focusing in the sense of sharpening. This approach was implemented in practice with application of the ImageTools graphical information processing library of the Maple package [31].

According to the conditions of the problem, dependence of the number of pixels Np in the phase path image on the value of the control parameter F was plotted (Fig. 5). It follows from the graph that the minimum number of pixels of the image is achieved at a critical value of the control parameter $F_0=2.556$. This value was clarified by reducing the interval containing F_0.

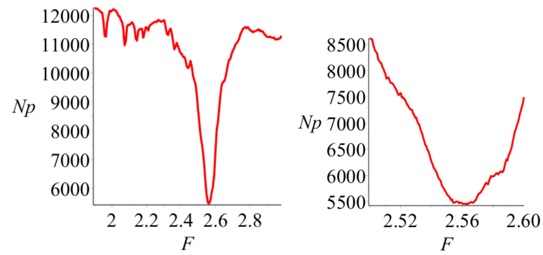


Fig. 5. Dependence of the number of pixels Np in the phase path image on F value: for $1.9 \leq F \leq 2.9$ (*a*); for $2.5 \leq F \leq 2.6$ (*b*)

After calculation of F_0, it is necessary to substitute its value instead of F in the system of Lagrange equations of the second kind (3) and numerically solve this system by Runge-Kutta method with respect to the functions $u(t)$ and $v(t)$. A sequence of values (u_i, v_i) at $t=t_i$ (where $i=1... S$) is obtained. To construct the path of movement of the second load in Oxy plane, it is necessary to put sequence of values (u_i, v_i) in expression (1) of virtual coordinates (x_2, y_2). The resulting adjacent points should be connected to a broken line. As a result, an approximate image of the path of movement of the second load in Oxy plane is found.

To implement this idea, a program was developed for building a phase path as an orthogonal projection of an integral line from a phase space to a phase coordinate plane with simultaneous computation of critical value of the control parameter with further definition of the path of the second load movement. The time of integration of the system of equations was determined. It will correspond to the minimum time in which the second load returns to its original position.

Obtained solutions to the problem. Let the lengths of the dual pendulum links be $d_1=1.5$ and $d_2=1$. Then, periodic paths of the second load of the pendulum will depend on the ratio of weight values of loads according to the following cases.

Case 1. For arbitrary identical values of weights $m_1=m_2$ at $F=2.556$, periodic paths of the second load close by their geometric forms are obtained. Their view is shown in Fig. 6.

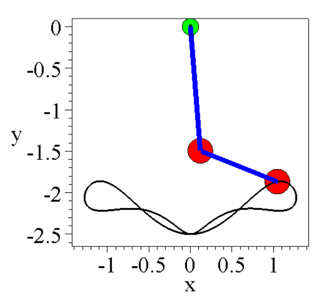
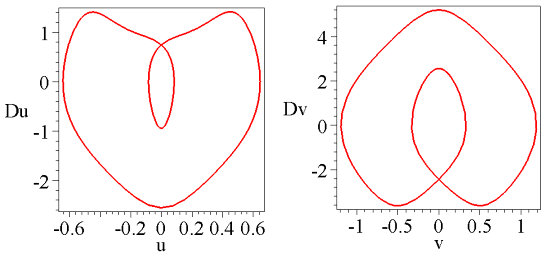


Fig. 6. The view of path of the second load at a value of $F=2.556$ in cases when $m_1=m_2$

The phase paths of the functions of generalized coordinates are shown in Fig. 7. Integration time $T \approx 3$.



a *b*

Fig. 7. The view of phase paths of the functions of generalized coordinates for case 1 at $m_1=m_2$ and F=2.556: for $u(t)$ (*a*); for $v(t)$ (*b*)

Case 2. When $m_1=km_2$, then periodic paths of the second load close in their geometric form are obtained. They are shown in Fig. 8. In this case, F values are taken from Table 1.

Table 1

Parameter values to provide a periodic path in the case of $m_1=km_2$

Value of k	2	3	4	5	6
Value of F	3.752	3.388	3.188	3.052	2.964

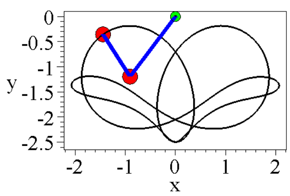
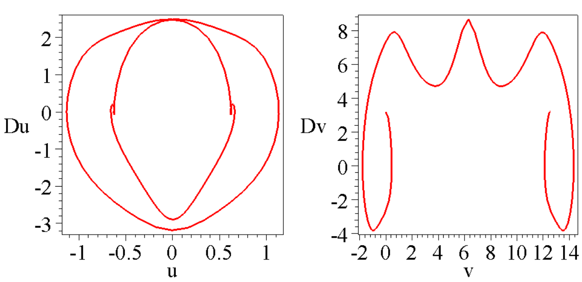


Fig. 8. The view of path of the second load in the case when $m_1=km_2$

The view of phase paths of the functions of generalized coordinates for case 2 is shown in Fig. 9. Integration time T≈5.4.



a *b*

Fig. 9. The view of phase paths of functions of generalized coordinates for the case at $m_1=km_2$ and the value of F taken from Table 1: for $u(t)$ (*a*); for $v(t)$ (*b*)

Case 3. When $km_1=m_2$, periodic paths of the second load close by their geometric form are obtained. They are shown in Fig. 10. The values of F are taken from Table 2.

Table 2

Values of parameters to provide a periodic path in the case of $km_1=m_2$

Value of k	2	3	4	5	6
Value of F	4.844	6.32	7.36	8.3	9.2

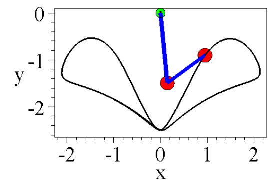
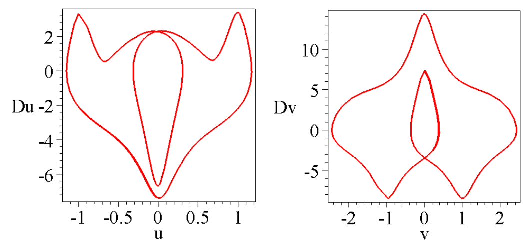


Fig. 10. The view of the second load path in the case when $km_1=m_2$

The view of phase paths of functions of generalized coordinates for case 3 is shown in Fig. 11. Integration time T≈3.4.



a *b*

Fig. 11. The view of phase paths of the functions of generalized coordinates for the case when $km_1=m_2$ and the value of F taken from Table 2: for $u(t)$ (*a*); for $v(t)$ (*b*)

The obtained phase paths enable determination of the ranges of change in the angle values as well as rates of this change in the values of angles.

4. 2. Calculation of periodic paths of movement of a swinging spring load

Here is a way to determine path of movement of the swinging spring load in vertical plane Oxy depending on the load weight, initial length of the spring in the unloaded state, stiffness of the spring and initial conditions for occurrence of oscillations.

Conditions set for idealization of oscillations of the swinging spring:

– oscillation takes place in a vertical plane enveloping the fixing (suspension) point;

– the load weight is concentrated in one point located in the spring axis from the nonfixed end;

– the spring is weightless and its axis remains straight during oscillation;

– supports in the nodes and the air resistance are absent during oscillation;

– the process of energy dissipation is slow in comparison with the characteristic time scales (the oscillatory system is conservative);

– parameters and initial conditions are given in conditional numerical units.

The swinging spring diagram is shown in Fig. 12.

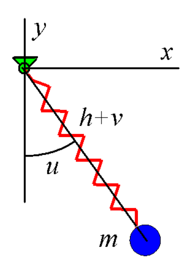


Fig. 12. The swinging spring diagram

Take the value of the angle formed by the axis of the swinging spring with the vertical axis Oy as the first generalized coordinate function $u(t)$. Relate the second generalized coordinate function $v(t)$ to longitudinal variation of the spring length in time; denote length of the swinging spring in unloaded state through h. Then virtual coordinates of the moving point load can be calculated according to the formulas:

$$x=(h+v)\sin u;$$

$$y=-(h+v)\cos u. \tag{5}$$

Set Lagrangian as a difference between kinetic and potential energies (g=9.81):

$$L=0.5m\left(\left(\frac{dv}{dt}\right)^2+(h+v)\left(\frac{du}{dt}\right)^2\right)-$$
$$0.5kv^2-mg(h+v)(1-\cos u)-mgv. \tag{6}$$

To form a system of Lagrange differential equations of the second kind, use the following relation (the point means time derivative):

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{u}}\right)-\frac{\partial L}{\partial u}=0;$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{v}}\right)-\frac{\partial L}{\partial v}=0. \tag{7}$$

As a result, the system of Lagrange equations of the second kind is obtained in the form:

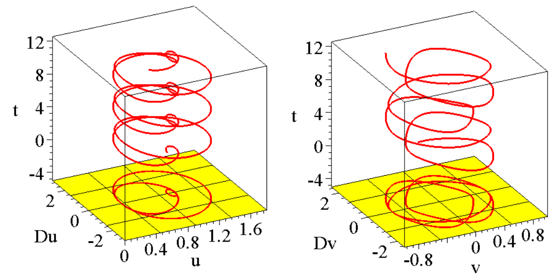
$$(v+h)\frac{d^2u}{dt^2}+2\frac{dv}{dt}\frac{du}{dt}+g\sin u=0; \tag{8}$$

$$\frac{d^2v}{dt^2}-(v+h)\left(\frac{du}{dt}\right)^2+\frac{kv}{m}-g\cos u=0.$$

The problem statement. Determine the value of weight m which would provide a periodic path of movement of load of the swinging spring with stiffness k and length h in unloaded state. In its initial position, the swinging spring is positioned vertically, that is, $u(0)=0$. Oscillation is initiated by means of an impulse applied to the spring load in direction of the Ox axis: $du(0)=1.5$. The value of 1.5 can be considered the initial rate of change in time of the angle u. Take initial values for parameter v of the spring extension as $v(0)=1$; $dv(0)=0$.

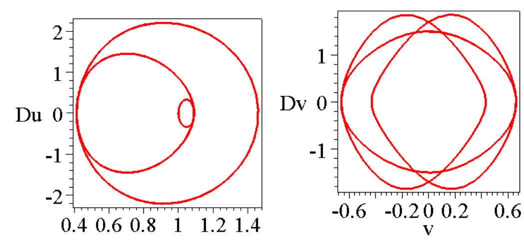
Solve the system of equations (8) with initial conditions $u(0)=0$; $du(0)=1.5$; $v(0)=1$; $dv(0)=0$ by applying numerical Runge-Kutta method. Choice of parameters m, k, and h ensures periodicity of the swinging spring load path.

Example 1. Let k=30 and h=1. Take weight m as controlling parameter of the swinging spring oscillation. Fig. 13 shows integral curves in phase spaces $\{u, Du, t\}$ and $\{v, Dv, t\}$ for the found critical value m=3.332. As a result, phase paths in $\{u, Du\}$ and $\{v, Dv\}$ planes are obtained (Fig. 14). With their help, it is possible to determine ranges of angle variation and rate of this variation (coordinate function $u(t)$) during oscillation of the swinging spring as well as elongation of the spring and elongation rate (coordinate function $v(t)$).



a *b*

Fig. 13. Integral curves for critical value m=3.332 in phase spaces: $\{u, Du, t\}$ (a); $\{v, Dv, t\}$ (b)



a *b*

Fig. 14. Phase paths in planes $\{u, Du\}$ and $\{u, Dv\}$ for: coordinate function $u(t)$ (a); coordinate function $v(t)$ (b)

To confirm value of the critical value m=3.332 found, use the graph of saturation of the phase path line image. Fig. 15 shows the graph of dependence of the number of pixels Np in the image of the phase path on the value of the control parameter m. Minimum number of image pixels is achieved at a critical value of the control parameter m_0=3.332.

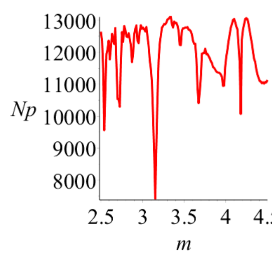


Fig. 15. The graph of dependence of the number of pixels Np in the image of the phase path on the m value

Following calculation of m_0, it is necessary to put its value into the place of m in the system of Lagrange equations of the second kind (8) and numerically solve it by the Runge-Kutta method with respect to $u(t)$ and $v(t)$ functions. A sequence of values of (u_i, v_i) at $t=t_i$ (where i=1... S) is obtained. To construct the path of movement of the second load in Oxy plane, it is necessary to put sequence of values (u_i, v_i). in expressions (5) of virtual coordinates (x, y). The resulting points should be connected to a broken line. As a result, an approximated image of the path of movement of the swinging spring load in Oxy plane is found (Fig. 16).

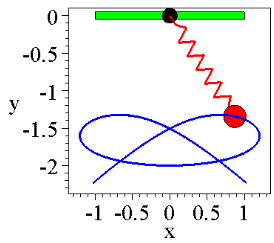
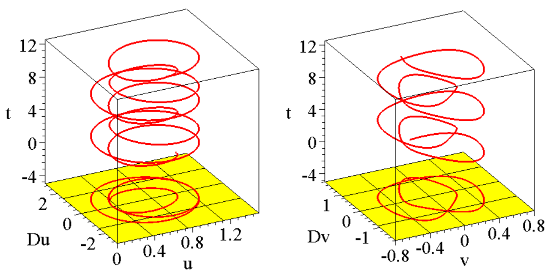


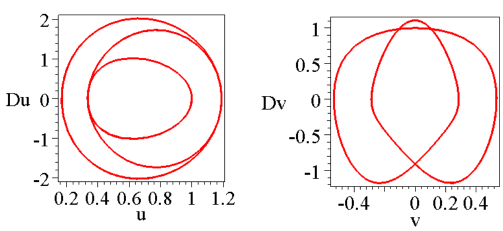
Fig. 16. The path of movement the swinging spring load for example 1

Example 2. Let m=1 and h=1. Select the value of stiffness k as a control parameter. Fig. 17 shows integral curves in phase spaces {u, Du, t} and {v, Dv, t} for the found critical value k=14.4. Fig. 18 shows phase paths of the corresponding generalized coordinate functions by means of which it is possible to determine ranges of their changes.



a *b*

Fig. 17. Integral curves for the critical value of k=14.4 in phase spaces: {u, Du, t} (*a*); {v, Dv, t} (*b*)



a *b*

Fig. 18. The phase paths in planes {u, Du} and {v, Dv}: coordinate function $u(t)$; coordinate function $v(t)$ (*b*)

To confirm the value of k=14.4, use the saturation graph of the phase path line (Fig. 19). Minimum number of image pixels is achieved at critical value of the control parameter k_0=14.4.

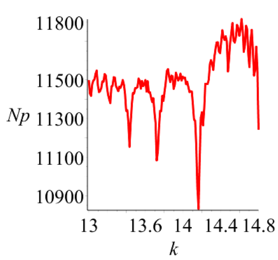


Fig. 19. The graph of dependence of the number of pixels Np in the phase path image on the m value

Following calculation of k_0=14.4, it is necessary to substitute its value for k in the system of Lagrange equations of the second kind (8) and numerically solve it by Runge-Kutta method with respect to the functions $u(t)$ and $v(t)$. A sequence of values of (u_i, v_i) is obtained at t=t_i (where i=1... S). To construct the path of movement of the swinging spring load in the Oxy plane, it is necessary to put the sequence of values (u_i, v_i) into expressions (5) of virtual coordinates (x, y). The resulting points should be connected to a broken line. As a result, an approximated image of the path of movement of the swinging spring load in the Oxy plane is found (Fig. 20).

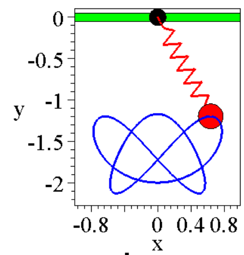
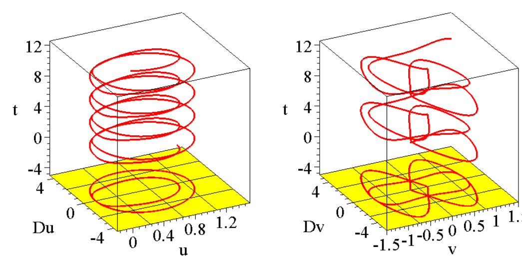


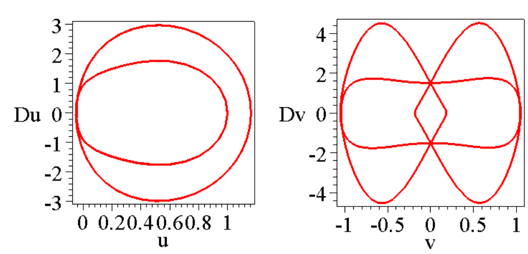
Fig. 20. The path of movement of the swinging spring load for example 2

Example 3. Let m=1 and k=10. Take the length h of the swinging spring without load as a control parameter. Fig. 21 shows integral curves in the phase spaces {u, Du, t} and {v, Dv, t} for the found critical value of h=0.39. Fig. 22 shows the phase paths of the corresponding generalized coordinate functions. With their help, it is possible to determine variation ranges.



a *b*

Fig. 21. Integral curves for the critical value of h=0.39 in phase spaces: {u, Du, t} (*a*); {v, Dv, t} (*b*)



a *b*

Fig. 22. The phase paths in planes {u, Du} and {v, Dv}: coordinate function $u(t)$; coordinate function $v(t)$

To confirm value of h=0.39, use the graph of saturation of the phase path line image (Fig. 23). Minimum number of the image pixels is achieved at a critical value of the control parameter h_0=0.39.

Following calculation of h_0=0.39, it is necessary to substitute its value in place of h in the system of Lagrange equations of the second kind (8) and numerically solve it by Runge-Kutta method with respect to the functions $u(t)$ and

$v(t)$. A sequence of values of (u_i, v_i) at $t=t_i$ (where i=1... S) is obtained. To construct the path of movement of the swinging spring load in the Oxy plane, it is necessary to put the sequence of values of (u_i, v_i) into expressions (5) of virtual coordinates (x, y), The resulting points should be connected to a broken line. As a result, an approximate image of periodic path of movement of the swinging spring load in the Oxy plane is found (Fig. 24).

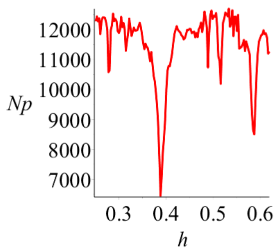


Fig. 23. The graph of dependence of the number of pixels Np in the phase path image on the value of h

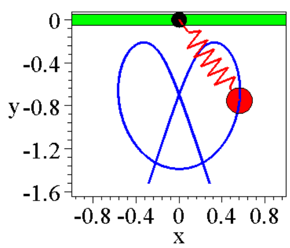


Fig. 24. The paths of the spring load movement for example 3

Consequently, the critical values of the control parameter found by the method of projection focusing can be confirmed using the graph of saturation of the image of the phase path line, that is, the graph of dependence of the number of pixels Np in the image of the phase path on the control parameter value.

4. 3. Calculation of periodic paths of movement of a load attached to a swinging spring with a moving suspension point

The method of determining the path of movement of a swinging spring load in a vertical plane was presented for the case when the point of its suspension is movable. Next, a description of the law of suspension point movement in a form of function $f(t)$ should be added to previously considered parameters (load weight, initial length of the spring in unloaded condition, spring stiffness and initial conditions for occurrence of oscillation). Note that in the case of movable point of suspension, one can expect not only strict periodic paths of the swinging spring load but also conditionally periodic paths because of significant nonlinearity of the oscillatory system, that is, such paths of load movement that will not go beyond boundaries of a certain band in the Oxy plane.

To describe oscillations of the swinging spring, take the value of the angle formed by the axis of the swinging spring and vertical Oy axis as the first generalized coordinate function $u(t)$. Relate another generalized coordinate function $v(t)$ to longitudinal variation of the spring in time and denote the length of the swinging spring in unloaded state through h (Fig. 2).

This study considers two cases of movement of the suspension point: along Ox and Oy axes.

Case 1. Let the point of suspension of the swinging spring move along the Ox axis according to law $x=f(t)$. Then virtual coordinates of the moving point load can be calculated by formulas (5). Lagrangian is taken as a difference between kinetic and potential energies (g=9.81):

$$L=0.5m\left(\left(\frac{df}{dt}\right)^2+\left(\frac{dv}{dt}\right)^2+(h+v)\left(\frac{du}{dt}\right)^2\right)+$$
$$+m\frac{df}{dt}\left(\frac{dv}{dt}\sin u+(h+v)\frac{du}{dt}\sin u\right)-$$
$$-0.5kv^2+mg(h+v)\cos u. \tag{9}$$

To set up a system of differential Lagrange equations of the second kind, use relation (7). As a result, the system of Lagrange equations of the second kind is obtained in this form:

$$\frac{d^2v}{dt^2}+\frac{d^2f}{dt^2}\sin u-$$
$$-(h+u)\left(\frac{du}{dt}\right)^2+kv-g\cos u=0; \tag{10}$$
$$(h+v)\frac{d^2u}{dt^2}-\left(\frac{df}{dt}\right)^2\cos u+$$
$$+2\frac{du}{dt}\frac{dv}{dt}+g\sin u=0.$$

Example 4. Determine the value of weight m which would provide a periodic path of movement of the load of the swinging spring with stiffness k and length h in unloaded state. In initial position, the swinging spring is positioned vertically, that is, $u(0)=0$. Oscillation is initiated by means of an impulse applied to the spring load in the direction of Ox axis: $du(0)=1$. This value can be considered as initial rate of variation in time of the angle u. Initial values for the v parameter of the spring extension have form $v(0)=2$; $dv(0)=0$. Let k=50 and h=2. Set the law of movement of the fixing point by function $f(t)=\sin(2t)$. Take the value of load weight m as a controlling parameter of the swinging spring oscillation.

Solve the system of equations (8) with initial conditions $u(0)=0$; $du(0)=1$; $v(0)=2$; $dv(0)=0$ using numerical Runge-Kutta method. Fig. 25 shows integral curves in phase spaces {u, Du, t} and {v, Dv, t} for the found critical value of m=5.142. Integration time T=16. Fig. 26 shows phase paths of corresponding generalized coordinate functions. With their help, it is possible to determine variation ranges. It can be seen that the phase paths cannot be "focused" as in the previous examples. Therefore, to maintain correctness, the further obtained paths of movement of the swinging spring load will be considered conditionally periodic.

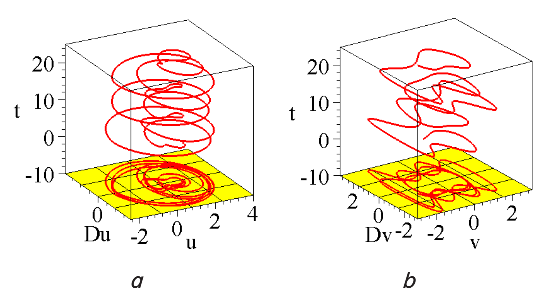
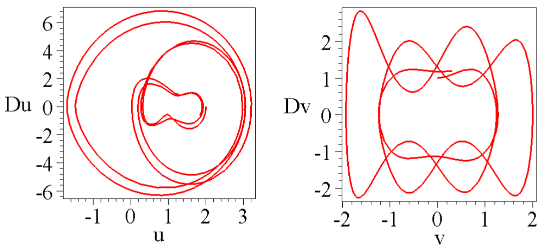


Fig. 25. Integral curves for a critical value of m=5.142 in phase spaces: {u, Du, t} (a); {v, Dv, t} (b)



a *b*

Fig. 26. Phase paths in planes {*u*, *Du*} and {*v*, *Dv*}: coordinate function *u*(*t*) (*a*); coordinate function *v*(*t*) (*b*)

To confirm the value of m=5.142, use the graph of saturation of image of the phase path line (Fig. 27). The minimum number of image pixels is achieved at a critical value of the control parameter, m_0=5.148.

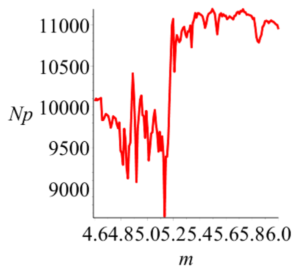


Fig. 27. The graph of dependence of the number of pixels *Np* in the image of the phase path on the *m* value

Following calculation of m_0=5.142, it is necessary to substitute its value into the place of m in the system of Lagrange equations of the second kind (8) and numerically solve it by Runge-Kutta method with respect to the functions $u(t)$ and $v(t)$. A sequence of values of (u_i, v_i) is obtained at t=t_i (where i=1... S). To construct the path of movement of the swinging spring load in the *Oxy* plane, it is necessary to put the sequence of values (u_i, v_i) into expressions (5) of virtual coordinates (x, y). The resulting points should be connected to a broken line. As a result, an approximated image of the periodic path of the swinging spring load movement in the *Oxy* plane is found for case 1 (Fig. 28). Since the phase paths have failed to be "focused" as in the previous examples, the resulting path of movement of the swinging spring load will be considered conditionally periodic. Visual analyzer has confirmed naturality of oscillations of the swinging spring with a moving suspension point which can be seen from the computer animations on the web site [32].

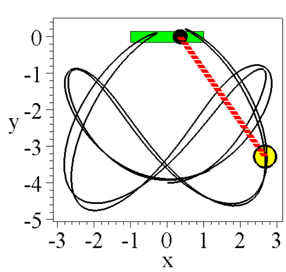
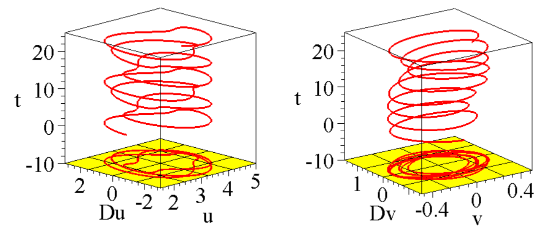


Fig. 28. The path of movement of the swinging spring load for example 4

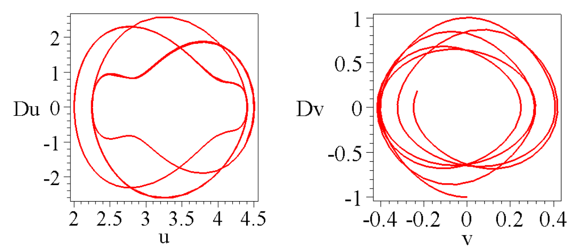
Example 5. Let us change direction of the impulse action to initiate movement of the swinging spring to the opposite, that is, take $du(0)$=–1. Solve the system of equations (8) with initial conditions $u(0)$=0; $du(0)$=–1; $v(0)$=2; $dv(0)$=0 by numerical Runge-Kutta method.

Fig. 29 shows integral curves in phase spaces {u, Du, t} and {v, Dv, t} for the found critical value m=16.571. Integration time T=16.7. Fig. 30 shows phase paths of the corresponding generalized coordinate functions with the help of which it is possible to determine their variation ranges.



a *b*

Fig. 29. Integral curves for a critical value of *m*=16.571 in phase spaces: {*u, Du, t*} (*a*); {*v, Dv, t*} (*b*)



a *b*

Fig. 30. Phase paths in planes {*u*, *Du*} and {*v*, *Dv*}: coordinate function *u*(*t*); coordinate function *v*(*t*) (*b*)

To confirm the value of m=16.571, use the graph of saturation of the phase path line image (Fig. 31). Minimum number of image pixels is achieved at a critical value of the control parameter m_0=16.571.

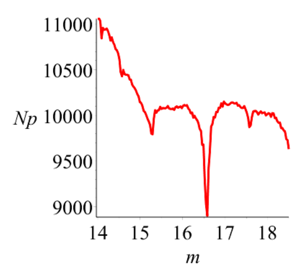


Fig. 31. The graph of dependence of the number of pixels *Np* in the image of the phase path on the value of *m*

Following calculation of m_0=16.571, it is necessary to substitute its value into the place of m in the system of Lagrange equations of the second kind (8) and numerically solve it by Runge-Kutta method with respect to the functions $u(t)$ and $v(t)$. A sequence of values of (u_i, v_i) at t=t_i (where i=1... S) is obtained. To construct the path of movement of the swinging spring load in the *Oxy* plane, put the sequence of values of (u_i, v_i) into expressions (5) of virtual coordinates (x, y). The resulting points should be connected to a broken line. As a result, an approximated image of the periodic path of movement of the swinging spring load in the *Oxy* plane was found for case 1 (Fig. 32).

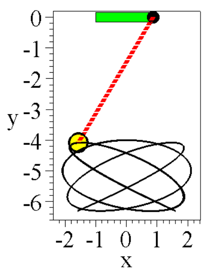


Fig. 32. The path of movement of the spring load for example 5

Thus, it can be stated that a periodic or conditionally periodic path can be obtained by changing direction of impulse action for initiating movement of the swinging spring from $du(0)=1$ to the opposite one: $du(0)=-1$.

Case 2. Let the point of suspension of the swinging spring move along the Oy axis by the law $y=f(t)$. Then virtual coordinates of the moving point load can be calculated by formulas (5). Set Lagrangian as a difference between kinetic and potential energies (g=9.81):

$$L=0.5m\left(\left(\frac{dv}{dt}\right)^2+v^2\left(\frac{du}{dt}\right)^2\right)-$$
$$-0.5k(v-h)^2-m\left(g+\frac{d^2f}{dt^2}\right)v\cos u. \quad (11)$$

To form a system of Lagrange differential equations of the second kind, relation (7) should be used. As a result, the system of Lagrange equations of the second kind is obtained in the form:

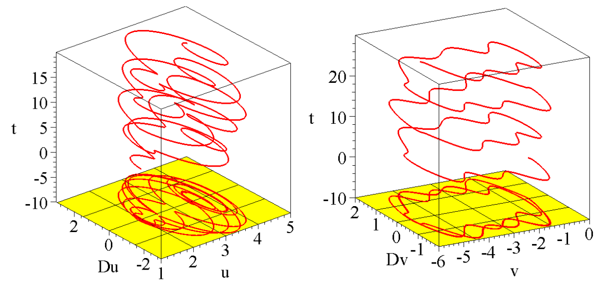
$$2m\frac{d^2v}{dt^2}+m\frac{d^2f}{dt^2}\cos u-2mv\left(\frac{du}{dt}\right)^2+$$
$$+k(v-h)+mg\cos u=0; \quad (12)$$

$$-2v\frac{d^2u}{dt^2}+\frac{d^2f}{dt^2}\sin u-4\frac{du}{dt}\frac{dv}{dt}+g\sin u=0.$$

Determine value of the weight m which would provide periodic path of movement of the load of the swinging spring with stiffness k and length h in unloaded state.

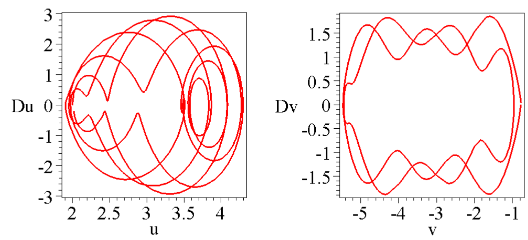
Example 6. Let the initial position of the swinging spring be determined by the angle $-\pi/4$, that is, $u(0)=-\pi/4$. The rate of variation of the angle value $du(0)=0$. Initial values for the parameter v of the spring extension are of the form $v(0)=2$; $dv(0)=0$. Take $k=450$ and $h=2.5$. Set the law of movement of the fixing point by function $y=0.5\cos(4t)$. Take the value of the load weight as a controlling parameter of the swinging spring oscillation.

Solve the system of equations (12) by numerical Runge-Kutta method with initial conditions $u(0)=-\pi/4$; $du(0)=0$; $v(0)=2$; $dv(0)=0$. Fig. 33 shows integral curves in the phase spaces $\{u, Du, t\}$ and $\{v, Dv, t\}$ for the found critical value of $m=22.57$. Integration time $T=17.2$. Fig. 34 shows the phase paths of corresponding generalized coordinate functions. With their help, it is possible to determine their variation ranges. Phase paths cannot be "focused" like in the previous examples. Therefore, the path of movement of the spring load is considered conditionally periodic.



a *b*

Fig. 33. Integral curves for a critical value of m=22.57 in phase spaces: $\{u, Du, t\}$ (*a*); $\{v, Dv, t\}$ (*b*)



a *b*

Fig. 34. Phase paths in $\{u, Du\}$ and $\{v, Dv\}$: coordinate function $u(t)$ (*a*); coordinate function $v(t)$ (*b*)

To confirm the value of m=22.57, use the graph of saturation of the phase path line image (Fig. 35). Minimum number of image pixels is achieved at a critical value of the control parameter m_0=22.57.

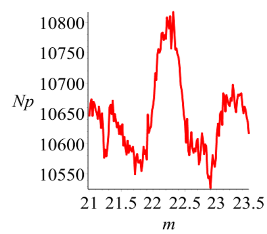


Fig. 35. The graph of dependence of the number of pixels *Np* in the image of the phase path on the value of *m*

Following calculation of m_0=22.57, it is necessary to put its value into the place of m in the system of Lagrange equations of the second kind (8) and numerically solve it by the Runge-Kutta method with respect to the functions $u(t)$ and $v(t)$. A sequence of values of (u_i, v_i) at $t=t_i$ (where i=1... S) is obtained. To construct the path of movement of the swinging spring load in the Oxy plane, put the sequence of values of (u_i, v_i) into expressions of (5) of virtual coordinates (x, y). The resulting points should be connected to a broken line. As a result, an approximated image of periodic path of movement of the swinging spring load in the Oxy plane is found for case 2 (Fig. 36).

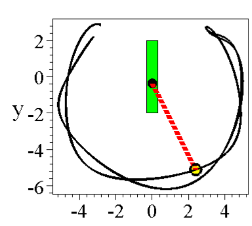


Fig. 36. The path of movement of the swinging spring load for example 6

Example 7. Let us consider another variant when the swinging spring is located at initial position at an angle $\pi/3$, that is, $u(0)=\pi/3$. The rate of angle variation: $du(0)=0$. Initial values for the parameter v of the spring extension: $v(0)=2$; $dv(0)=0$. Let k=50 and h=2.5. Set the law of movement of the point of attachment by function $y=\cos(3t)$. Take the value of the load weight as a controlling parameter of oscillation of the swinging spring.

Solve the system of equations (12) by numerical Runge-Kutta method with initial conditions $u(0)=\pi/3$; $du(0)=1$; $v(0)=2$; $dv(0)=0$. Fig. 37 shows integral curves in phase spaces $\{u, Du, t\}$ and $\{v, Dv, t\}$ for the found critical value of m=5.7557. Integration time T=25.3. Fig. 38 shows phase paths of the corresponding generalized coordinate functions with the help of which it is possible to determine their variation ranges. It is seen that the phase paths cannot by "focused" as in the previous examples. Therefore, it is necessary to expect the path of movement of the swinging spring load to be conditionally periodic.

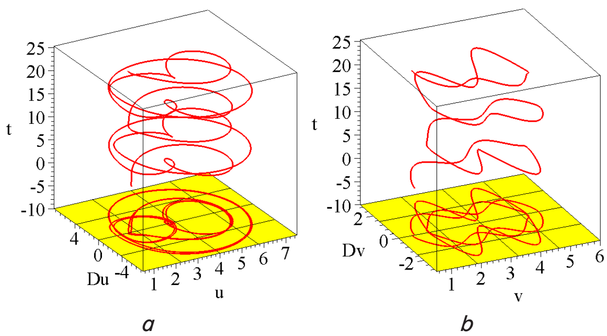


Fig. 37. Integral curves for critical value of m=5.7557 in phase spaces: $\{u, Du, t\}$ (*a*); $\{v, Dv, t\}$ (*b*)

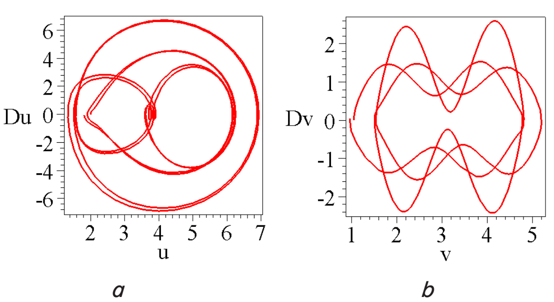


Fig. 38. The phase paths in planes $\{u, Du\}$ and $\{v, Dv\}$: coordinate function $u(t)$ (*a*); coordinate function $v(t)$ (*b*)

To confirm value of m=5.7557, the graph of saturation of the phase path line image can be used (Fig. 39). Minimum number of image pixels is achieved at a critical value of the control parameter m_0=5.7557.

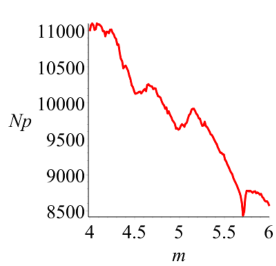


Fig. 39. The graph of dependence of the number of pixels *Np* in the image of the phase path on the *m* value

Following calculation of m_0=5.7557, it is necessary to substitute its value into the place of m in the system of Lagrange equations of the second kind (12) and numerically solve it by Runge-Kutta method with respect to the functions $u(t)$ and $v(t)$. A sequence of values of (u_i, v_i) are obtained at $t=t_i$ (where i=1... S). To construct the path of movement of the swinging spring load in the Oxy plane, it is necessary to put the sequence of values of (u_i, v_i). in expressions (5) of virtual coordinates (x, y). The resulting points should be connected to a broken line. As a result, an approximated image of the periodic path of movement of the swinging spring load in the Oxy plane is found for example 7 (Fig. 40).

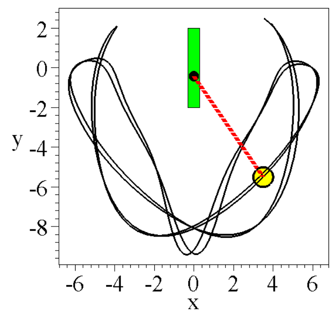


Fig. 40. The path of movement of the swinging spring load for example 7

The conditionally periodic paths obtained in this and preceding examples can be explained by substantial nonlinearity of the problem of oscillation of a swinging spring with a moving point of suspension. By involving visual analyzer in the process of visualization of oscillations through computer animation, one can make sure of natural character of oscillation of the swinging spring with a moving point of suspension. Confirmation for this fact can be found on the web site [32] where computer animations of oscillation of various swinging springs are provided.

5. Discussion of results obtained in computer simulation of the paths of movement of the swinging spring loads

The obtained results can be explained by the possibility of applying Lagrange variational principle to calculation of mechanical oscillations of the type of swinging spring oscillations. This has allowed us to use Lagrange equations of the second kind to describe movement of the spring load.

Consideration of the ratio $\frac{mg}{kl}=\frac{1}{4}$ for the cases of a wide range of variation of the parameter values belongs to the not yet realized possibilities of study of the swinging spring oscillation. Here, m is the load weight, k is the spring stiffness, l is the spring length in unloaded state, g=9.81.

Under the condition of fulfillment of this correlation between parameters of the vibrational system, angular swinging of the spring is most effectively performed at the expense of this spring energy. Development of random transverse perturbation will continue to a definite value of amplitude since energy reserves of the spring are finite. After reaching such an amplitude, stretching (or compression) of the spring occurs again in the course of oscillation of the swinging spring. This periodic repumping of the spring energy into energy of transverse oscillation of the load and back appears to be possible in a rather narrow range of variation of parameters with a maximum value corresponding to the indicated ratio. It is necessary to check under what conditions this relationship is executed with acceptable accuracy and how it affects the image of periodic paths of movement of the spring load. It is necessary to reveal number of possible periodic paths for a certain set of input parameters as well as classify images of periodic paths and perform their gradation taking into consideration growth of their lengths.

In addition, it is necessary to continue study in the direction of using the swinging spring as a model for studying nonlinear coupled systems. Indeed, three energy components similar to the spring and pendulum movements as well as the connection between them necessary for this process are identified for a swinging spring. This approach can be applied, in principle, to arbitrary nonlinear coupled systems to show how coupling mediates internal energy exchanges and how energy distribution varies according to the system parameters.

It will be interesting to investigate from these positions nonlinear coupled systems with interacting subsystems on examples of engineering problems. The first step to this goal will be the study of mechanical devices where springs will affect the path of oscillation of their loads. As examples, it is expedient to consider mechanisms with moving loads, the schemes having the form:

– two springs with a common load;

– a pendulum attached to a suspended spring;

– a pendulum has length influenced by the spring;

– a pendulum under a moving cart whose position is influenced by the spring;

– a load at the end of the spring suspended to a moving cart.

Difficulties in development of the studies in this direction will arise when trying to solve an inverse problem in the following statement. Let there be a curve having shape belonging to figures of Lissage class. It is necessary to select values of the swinging spring parameters (load weight, spring stiffness and length in unloaded condition) so that the path of the load movement is similar to the selected curve.

7. Conclusions

1. Among the *a priori* chaotic oscillations of a double pendulum, such oscillation was found when the second load moves in a periodic path. This has made it possible to extend the method of problem solution to the problems of determining periodic paths of movement of the spring load.

2. Variants of calculation for obtaining of a periodic path of a swinging spring load were given when the spring parameters are set:

– stiffness of the spring and its length without load at an unknown load weight (for example, h=1; m=3.332; k=40; v_0=1; Dv_0=0: u_0=0; Du_0=1.5; T=8.4);

– weight of the spring load and the spring length without load at an unknown spring stiffness (for example, h=1; m=1; k=14.4; v_0=1; Dv_0=0: u_0=0; Du_0=1; T=8.4);

– weight of the spring load and the spring stiffness at an unknown spring length without load (for example, h=0.39; m=2; k=40; v_0=1; Dv_0=0: u_0=0; Du_0=1.5; T=6).

3. Values of the parameters for providing a conditionally periodic path of movement of the point load of a swinging spring with a movable fixing point (for example, m=16.571; k=50; h=2; u_0=2; Du_0=0; v_0=0; Dv_0=–1; x=sin(2*t); T=16.7).

4. For each variant of calculation of the swinging spring, phase paths of the functions of generalized coordinates (values of angles of deviation and elongation) were constructed which has made it possible to estimate the range of variation of these quantities and rates of this variation.

5. Reliability of the obtained results was illustrated by computer animation of oscillations of corresponding swinging springs demonstrated at the Internet site [32], where, by involving visual analyzer, it is possible to verify natural character of oscillations of the swinging spring including moving point of suspension.

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